

STRUCTURI ALGEBRICE II

OVIDIU FURDUI

Teste GrILA de Matematică 2024

$$M = [5, 7]$$

$$x * y = xy - 6x - 6y + \alpha$$

67, 68, 69.

(67) M parte stabilă în raport cu operația $*$

$$\forall x, y \in M \Rightarrow x * y \in M \Rightarrow$$

$$5 \leq x * y \leq 7 \quad \forall x, y \in [5, 7] \Rightarrow$$

$$5 \leq xy - 6x - 6y + \alpha \leq 7$$

$$5 \leq (x-6)(y-6) + \alpha - 36 \leq 7 \quad \forall x, y \in [5, 7]$$

$$\begin{aligned} x-6 &= a \\ y-6 &= b \end{aligned} \Rightarrow \underline{-1 \leq a, b \leq 1}$$

$$5 \leq ab + \alpha - 36 \leq 7 \quad | \begin{matrix} - \\ -6 \end{matrix} \Rightarrow$$

$$-1 \leq ab + \alpha - 42 \leq 1 \rightarrow \begin{matrix} \downarrow -1 & & \downarrow 1 \\ 41 - \alpha \leq ab \leq 43 - \alpha \end{matrix}$$

$$ab = -1 \Rightarrow \alpha - 42 \geq 0$$

$$ab = 1 \Rightarrow \alpha - 42 \leq 0 \Rightarrow \boxed{\alpha = 42} \Rightarrow \textcircled{+}$$

(68) În monoidul $(M, *)$ elementul neutru este:

$$M \times M \rightarrow M, (x, y) \rightarrow x * y \in M$$

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$$z * y = zy - 6x - 6y + 42$$

$$\alpha = 42$$

$$x * e = e * x = x \quad \forall x \in [5, 7]$$

$$x * e = xe - 6x - 6e + 42 = x$$

$$\underbrace{xe - 7x - 6e + 42}_{\underbrace{\quad}} = 0,$$

$$(x-6)e - 7(x-6) = 0 \Rightarrow (x-6)(e-7) = 0.$$
$$\forall x \in [5, 7]$$

$$\Rightarrow \boxed{e=7} \Rightarrow \textcircled{A}$$

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$(M, *)$ monoid.

elemente simetrizabile

$x \in [5, 7]$ simetrizabil $\exists x' \in [5, 7]$

$$x * x' = x' * x = e = 7$$

$$x * x' = xx' - 6x - 6x' + 42 = 7 \quad | -6$$

$$\underbrace{xx' - 6x - 6x' + 36}_{\underbrace{\quad}} = 1$$

$$x(x'-6) - 6(x'-6) = 1 \Rightarrow$$

$$(x-6)(x'-6) = 1.$$

$$x, x' \in [5, 7] \Rightarrow x-6, x'-6 \in [-1, 1].$$

$$-1 \leq x-6, x'-6 \leq 1 \Rightarrow (x-6)(x'-6) = 1 \Rightarrow$$

$$x-6 = 1 \text{ si } x'-6 = 1 \text{ SAU } x-6 = -1 \text{ si } x'-6 = -1$$

$$\begin{array}{l}
 x=7 \\
 x'=7
 \end{array}
 \text{ sau }
 \begin{array}{l}
 x=5 \\
 x'=5
 \end{array}
 \begin{array}{l}
 7'=7 \\
 5'=5
 \end{array}$$

$$7 * 7 = 7 \cdot 7 - \cancel{6 \cdot 7} - \cancel{6 \cdot 7} + 42 = 49 - 42 = 7 = e$$

$$7' = 7$$

$$5 * 5 = 5 \cdot 5 - 6 \cdot 5 - 6 \cdot 5 + 42 = -5 - 30 + 42 = 7$$

$$5' = 5$$

$$\{5, 7\} \Rightarrow \textcircled{C}$$

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$$\mathbb{Z} * \mathbb{Z}$$

$$(x, y) * (a, b) = (xa, xb + ya)$$

element neutru (e_1, e_2)

$$(x, y) * (e_1, e_2) = (e_1, e_2) * (x, y) = (x, y)$$

$$\forall (x, y) \in \mathbb{Z} \times \mathbb{Z}$$

* - comutativa

$$(x, y) * (e_1, e_2) = (x, y) \quad \forall x, y \in \mathbb{Z}$$

$$(xe_1, xe_2 + ye_1) = (x, y) \Rightarrow$$

$$xe_1 = x \quad \Rightarrow \quad x(e_1 - 1) = 0 \quad \forall x \in \mathbb{Z}$$

$$xe_2 + ye_1 = y \quad \Rightarrow \quad xe_2 + ye_1 = y \quad \forall x, y \in \mathbb{Z}$$

$$e_1 = 1 \quad \Rightarrow \quad xe_2 = 0 \Rightarrow e_2 = 0$$

$$(e_1, e_2) = (1, 0) \Rightarrow \textcircled{B}$$

verificare!

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$$(72) \quad x * y = \frac{x-y}{1-xy}; \quad x, y \in (-1, 1)$$

element neutru = ?

$$\boxed{x * e = e * x = x} \quad \forall x \in (-1, 1)$$

$$x * e = x \Rightarrow \frac{x-e}{1-xe} = x \Rightarrow$$

$$x - e = x - x^2 e \Rightarrow 0 = e(1 - x^2) \Rightarrow e = 0.$$

$|x| < 1 \quad 1 - x^2 \neq 0.$

Verific $e * x = x.$ $0 * x = x.$

$$0 * x = \frac{0-x}{1-0 \cdot x} = \frac{-x}{1} = -x \neq x.$$

ATENȚIE Legea * nu e comutativă.

$$x * y = \frac{x-y}{1-xy} \neq y * x = \frac{y-x}{1-yx} = -\frac{x-y}{1-xy} = -x * y$$

elem. neutru 1 NU EXISTĂ \textcircled{B}

76, 73, 74, 75 - date

- calcul direct

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$$\forall x, y \in \mathbb{R} \quad x * y = xy - ax + by$$

$$a, b \in \mathbb{R}$$

$(\mathbb{R}, *)$ Monoids $a, b \in ?$

Monoid $\begin{cases} \text{legea asociativitatii} \\ \exists \text{ elem. neutru} \end{cases}$

• asociativitate

$$(x * y) * z = x * (y * z) \quad \forall x, y, z \in \mathbb{R}$$

$$\begin{aligned} (x * y) * z &= (xy - ax + by) * z \\ &= (xy - ax + by) \cdot z - a(xy - ax + by) + bz \\ &= \boxed{xyz - axz + byz - axy + a^2x - aby + bz} \end{aligned}$$

$$\begin{aligned} x * (y * z) &= x * (yz - ay + bz) \\ &= x(\cancel{yz} - ay + bz) - ax + b(yz - ay + bz) \\ &= \boxed{xyz - axy + bxz - ax + byz - aby + b^2z} \end{aligned}$$

$$-axz + a^2x + bz = bxz - ax + b^2z$$

$$(a+b)xz - (a+a^2)x + (b^2-b)z = 0 \quad \forall x, y, z \in \mathbb{R}$$

$$\Rightarrow \begin{cases} a+b=0. \\ a+a^2=0. \\ b^2-b=0. \end{cases}$$

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$$(a+b)xz - (a+a^2)x + (b^2-b)z = 0, \quad : xz \neq 0$$

$$a+b - (a+a^2) \cdot \frac{1}{z} + (b^2-b) \cdot \frac{1}{x} = 0$$

$$\forall x, z \neq 0$$

$$z, x \rightarrow \forall$$

$$z = x = 1 \rightarrow \forall \Rightarrow \underline{\underline{a+b=0}}$$

$$a+b=0$$

$$a(a+1)=0 \rightarrow a=0 \text{ sau } a=-1$$

$$b(b-1)=0 \rightarrow b=0 \text{ sau } b=1$$

$$\boxed{a=0, b=0}$$

$$\cancel{a=0}, \cancel{b=1}$$

$$a+b \neq 0$$

$$\cancel{a=-1}, \cancel{b=0}$$

$$a+b \neq 0$$

$$\boxed{a=-1, b=1}$$

$$\underline{a=b=0}$$

$$x * y = xy$$

elem. neutru

$$x * e = e * x = x \quad \forall x \in R$$

$$xe = x \Rightarrow x(e-1) = 0 \quad \forall x \in R$$

$$\boxed{e=1}$$

$$a=-1$$

$$\underline{b=1}$$

$$\Rightarrow x * y = xy + x + y$$

elem neutru

$$x * e = e * x = x \quad \forall x \in R$$

$$xe + x + e = x \Rightarrow e(x+1) = 0 \quad \forall x \in R \quad \boxed{e=0}$$

$$\boxed{0}$$

(194) (ϕ^*, \circ) și (\mathbb{R}^*, \cdot) grupuri

$a \in \mathbb{R}^*$ și $b \in \mathbb{R}$

$f: \phi^* \rightarrow \mathbb{R}^*$; $f(z) = a|z| + b$.
morfism de grupuri

$$f(z_1 \circ z_2) = f(z_1) \cdot f(z_2)$$

$$\forall z_1, z_2 \in \phi^*$$

$$a|z_1 z_2| + b = (a|z_1| + b)(a|z_2| + b)$$

$$\forall z_1, z_2 \in \phi^*$$

$$a|z_1 z_2| + b = a^2 |z_1 z_2| + ab|z_1| + ab|z_2| + b^2$$

$$(a^2 - a) |z_1 z_2| + ab|z_1| + ab|z_2| + b^2 - b = 0$$

$$\forall z_1, z_2 \in \phi^*$$

$$a^2 - a + ab \frac{1}{|z_2|} + ab \frac{1}{|z_1|} + \frac{b^2 - b}{|z_1 z_2|} = 0$$

$$z_1, z_2 \neq 0$$

$$z_1, z_2 = \infty \rightarrow \underline{a^2 - a = 0}$$

$$ab|z_1| + ab|z_2| + b^2 - b = 0 \quad ; \quad |z_1|$$

$$ab = 0 \Rightarrow b^2 - b = 0$$

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$$a^2 - a = 0.$$

$$\rightarrow a(a-1) = 0, \Rightarrow a = 1$$

$$ab = 0$$

$$a \neq 0.$$

$$\rightarrow b = 0.$$

$$b^2 - b = 0$$

$$a = 1 \text{ și } b = 0.$$

$$f(z) = |z|.$$

(1)

192 $(\mathbb{Z}[i], \cdot)$ monoid.

$$\mathbb{Z}[i] = \{a+ib; a, b \in \mathbb{Z}\}$$

elemente inversabile. ?

elem neutru : 1

$a+ib \in \mathbb{Z}[i]$ inversabil $\exists c+id \in \mathbb{Z}[i]$

$$a \cdot \checkmark \quad (a+ib)(c+id) = 1. \Rightarrow$$

$$\overline{(a+ib)(c+id)} = \overline{1} = 1$$

$$\Rightarrow (a-ib)(c-id) = 1$$

\Rightarrow
 \downarrow
inmultim

$$(a+ib)(a-ib)(c+id)(c-id) = 1$$

FORMULA $x, y \in \mathbb{R}$ $(x+iy)(x-iy) = x^2 + y^2$

Formula
echivalentă - 9 -
$$\boxed{z \in \phi, z \bar{z} = |z|^2}$$

$$(a^2 + b^2)(c^2 + d^2) = 1$$

$$a, b \in \mathbb{Z} \Rightarrow a^2 + b^2 \in \mathbb{Z}$$

$$c, d \in \mathbb{Z} \Rightarrow c^2 + d^2 \in \mathbb{Z}$$

} \Rightarrow

$$a^2 + b^2 = 1 \Rightarrow a^2 = 1, b^2 = 0 \text{ sau } a^2 = 0, b^2 = 1$$

$$c^2 + d^2 = 1 \Rightarrow c^2 = 1, d^2 = 0 \text{ sau } c^2 = 0, d^2 = 1$$

$$a^2 = 1 \Rightarrow a = \pm 1 \quad \neq 1$$

$$b^2 = 0 \Rightarrow b = 0.$$

$$a^2 = 0 \Rightarrow a = 0$$

$$b^2 = 1 \Rightarrow b = \pm 1 \quad \neq 1$$

$$c^2 = 1 \Rightarrow c = \pm 1$$

$$d^2 = 0 \Rightarrow d = 0 \quad \neq 1.$$

$$c^2 = 0 \Rightarrow c = 0$$

$$d^2 = 1 \Rightarrow d = \pm 1 \quad \neq 1$$

elemente inversabile $\{\pm 1, \pm i\}$ (B)
 $= \{-1, 1, -i, i\}$

verificare

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\mathbb{Z} . $x * y = xy + mx + my + a$

$m \in \mathbb{Z}$.

$a = ?$ $(\mathbb{Z}, *)$ - monoid.

- $\left\{ \begin{array}{l} * \text{ - associative} \\ \exists \text{ elem neutru} \end{array} \right.$

associativitate

$(x * y) * z = x * (y * z) \quad \forall x, y, z \in \mathbb{Z}$

$(x * y) * z = (xy + mx + my + a) * z$
 $= (xy + mx + my + a) * z + m(xy + mx + my + a)$
 $+ mz + a$

$= \boxed{xy z + mxz + myz + az + mxy + m^2x + m^2y + am + mz + a}$

$x * (y * z) = x * (yz + my + mz + a)$

$= \boxed{x \cdot yz + mxy + mxz + ax + mx + myz + m^2y + m^2z + ma + a}$

$\underline{az + m^2x + mz} = \underline{ax + mx + m^2z}$

$(a + m - m^2)z + (m^2 - a - m)x = 0 \quad \forall x, z \in \mathbb{Z}$

$(m - m^2 + a)(z * x) = 0. \quad \forall x, z \in \mathbb{Z}$

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$$m - m^2 + a = 0$$

$$\boxed{a = m^2 - m}$$

$$x * y = xy + mx + my + m^2 - m$$

elem neutru

$$x * e = e * x = x \quad \forall x \in \mathbb{Z}$$

$$x * e = xe + mx + me + m^2 - m = x$$

$$xe + mx - x + m(e + m - 1) = 0$$

$$x(e + m - 1) + m(e + m - 1) = 0$$

$$(x + m)(e + m - 1) = 0 \quad \forall x \in \mathbb{Z}$$

$$\boxed{e = 1 - m}$$

\boxed{E}

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$$\mathbb{R}_s \quad x * y = \sqrt[n]{x^n + y^n}$$

$$\boxed{(\mathbb{R}, *) \text{ grup.} \quad n = ?}$$

- * asociativ
- \exists elem neutru
- \forall elem este simetri'zabil.

lega asociativa

$$(x * y) * z = x * (y * z) \quad \forall x, y, z \in \mathbb{R}$$

$$(x * y) * z = \sqrt[n]{x^n + y^n + z^n} = x * (y * z)$$

elem. neutru

$$x * e = e * x = x \quad \forall x \in \mathbb{R}$$

$$x * e = x \Rightarrow \sqrt[n]{x^n + e^n} = x \Rightarrow x^n + e^n = x^n$$

$$\Rightarrow e^n = 0 \Rightarrow \underline{\underline{e = 0}}$$

VERIFICARE

$$x * 0 = \sqrt[n]{x^n + 0^n} = \sqrt[n]{x^n} = x$$

$$\sqrt[n]{x^n} = \begin{cases} x, & n \text{ impar} \\ \pm x, & n \text{ par.} \end{cases} \quad \left(\sqrt[n]{x^n} - |x| = \pm x \right)$$

$$\Rightarrow \boxed{n \text{ impar}}$$



$$\text{Simetric } x \Rightarrow -x,$$

$$x * (-x) = (-x) * x = 0,$$

$$x * x = \sqrt[n]{x^n + (-x)^n} = 0, \\ n \text{ impar,}$$

Metoda 2:

$$f: \mathbb{R} \rightarrow \mathbb{R}; f(x) = \sqrt[n]{x}$$

n impar $\Rightarrow f$ bijectivă.
 $n = 2k+1; k \in \mathbb{N}^*$

$$f^{-1}: \mathbb{R} \rightarrow \mathbb{R} \quad f^{-1}(u) = u^n;$$

$$(\mathbb{R}, +) \xrightarrow{f(x) = \sqrt[n]{x}} (\mathbb{R}, *)$$

$(\mathbb{R}, +)$ grup.

$f(x) = \sqrt[n]{x}$ - bijectivă \Rightarrow

induce pe \mathbb{R} operația "*" definită prin

$$f(x+y) = \sqrt[n]{f(x) * f(y)} \quad \begin{matrix} f(x) = u \Rightarrow \\ x = f^{-1}(u) \end{matrix}$$

$$\begin{aligned} u * v &= f(f^{-1}(u) + f^{-1}(v)) \\ &= f(u^n + v^n) = \sqrt[n]{u^n + v^n} \end{aligned}$$

$$u * v = \sqrt[n]{u^n + v^n} \quad u, v \in \mathbb{R}$$

$$(\mathbb{R}, +) \cong (\mathbb{R}, *) \quad u * v = \sqrt[n]{u^n + v^n};$$

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$(M_2(\mathbb{Z}), \cdot)$ MONOID.

elemente inversabile

$A \in M_2(\mathbb{Z})$ inversabil $\Rightarrow \exists B \in M_2(\mathbb{Z})$

$$AB = BA = I_2,$$

$$AB = I_2 \Rightarrow \det A \det B = 1 \Rightarrow \det A = \pm 1, \\ \det A, \det B \in \mathbb{Z}.$$

Observatie

$$A \in M_2(\mathbb{C})$$

$$A^2 - (\text{tr} A) A + (\det A) I_2 = 0_2.$$

$$\det A \neq 0 \Rightarrow A^2 - (\text{tr} A) A = -(\det A) I_2,$$

$$\Rightarrow A (A - (\text{tr} A) I_2) = -(\det A) I_2 \quad | : -\det A$$

$$A \left(\frac{A - (\text{tr} A) I_2}{-\det A} \right) = I_2 \Rightarrow \\ \underbrace{\hspace{10em}}_{A^{-1}}$$

$$A^{-1} = \frac{1}{\det A} \left[(\text{tr} A) I_2 - A \right] \quad \textcircled{E}$$

$$\det A = \pm 1 \Rightarrow A^{-1} = \pm 1 \left[(\text{tr} A) I_2 - A \right].$$

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$$f(x, y) = \frac{ax + by}{1 + xy}, \quad a, b \in \mathbb{R}$$

lege de compozitie pe $(-1, 1)$ dacă

$$(*) \quad \left[-1 < \frac{ax + by}{1 + xy} < 1 \quad \forall x, y \in (-1, 1) \right]$$

$$x=0, \quad -1 < by < 1 \quad \forall y \in (-1, 1)$$

$$y \rightarrow 1 \Rightarrow -1 \leq b \leq 1 \Rightarrow b \in [-1, 1].$$

$$y=0, \quad -1 < ax < 1 \quad \forall x \in (-1, 1)$$

$$x \rightarrow 1 \quad -1 \leq a \leq 1 \quad a \in [-1, 1].$$

$$y = -x \quad -1 < \frac{(a-b)x}{1-x^2} < 1 \quad \forall x \in (-1, 1).$$

$$-(1-x^2) < (a-b)x < 1-x^2 \quad \forall x \in (-1, 1)$$

$$x \rightarrow 1 \quad 0 \leq a-b \leq 0 \Rightarrow a=b$$

$$a=b \in [-1, 1]. \rightarrow \boxed{\Delta}$$

Verificare $-1 < a \cdot \frac{x+y}{1+xy} < 1 \quad \forall x, y \in (-1, 1)$

$$|a| \leq 1$$

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$$\left| a \cdot \frac{x+y}{1+xy} \right| = |a| \left| \frac{x+y}{1+xy} \right| \leq \left| \frac{x+y}{1+xy} \right| < 1$$

pt ca^v

$$-1 < \frac{x+y}{1+xy} < 1 \quad \forall x, y \in (-1, 1)$$

vezi lectia I.

sau calcul direct

$$-1 - xy < x+y < 1+xy \quad \forall x, y \in (-1, 1).$$

$$x+y < 1+xy \Rightarrow 1+xy - x - y > 0. \Rightarrow (1-x)(1-y) > 0.$$

$$-1 - xy < x+y \Rightarrow x+y + 1 + xy = (1+x)(1+y) > 0.$$